1. (20) Let C be a cycle in an connected undirected weighted graph $G = (V, E, w)$.

   (a) Let e be the unique heaviest edge in the cycle C. Show that e cannot be in any minimum spanning tree of G.

   (b) Let e be the unique lightest edge in the cycle C. Can we conclude that e must be in some minimum spanning tree of G? Justify your answer.

2. (30) Let $G = (V, E, w)$ be a connected weighted undirected graph. Assume that the weights of the edges are distinct. For each vertex $v \in V$, let $\alpha(v)$ be the edge with minimum weight among all edges incident at the vertex v. Let $E_\alpha = \bigcup_{v \in V} \alpha(v)$.

   (a) (10) Show that the induced subgraph $G[E_\alpha]$ contains no cycles.

   (b) (15) Show that there is a minimum spanning tree which contains all edges in E_α.

   (c) (5) Is the induced graph $G[E_\alpha]$ necessary a minimum spanning tree of G?

3. (20) Let $G = (V, E, w)$ be a connected weighted graph. Suppose that Dijkstra's algorithm is used to solve the single source shortest path problem on the weighted graph G.

   (a) Give an example to show that Dijkstra's algorithm does not work for graphs with negative weights.

   (b) Can we first add a constant value to the weight of each edge to make all weights non-negative, then run the Dijkstra's algorithm to compute shortest paths? Justify your answer.

4. (20) Let $G = (V, E, w)$ be a connected weighted undirected graph. Given a vertex $s \in V$ and a shortest path tree T_s with respect to the source s, design a linear time algorithm for checking whether the shortest path tree T_s is correct or not.

5. (20) A contiguous sub-sequence of a sequence $x_1, x_2, \ldots, x_n$ is a sub-sequence $x_i, x_{i+1}, x_{i+2}, \ldots, x_j$ for some $1 \leq i \leq j \leq n$. Given a sequence of integers $x_1, x_2, \ldots, x_n$, find a contiguous sub-sequence whose sum is maximized. Design a linear-time algorithm for the problem by using dynamic programming approach. That is, define the subproblems and write down the recurrence equation for solving the problem. Then use the input

   $$5, 15, -30, 10, -5, 40, 10$$

   to show how to compute the solution.

6. (40) Given a string of characters $s_1 s_2 \ldots s_n$. It is believed that the string is a document in which all space and punctuations have been removed. Reconstruct the document using a dictionary, which is available in the form of Boolean function $d(w) = 1$ if and only if w is a word. Use the example

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   wewillmeetatmidnight
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   to show how this problem can be solved by the following methods.

   (a) Constructed a graph from the string $s_1 s_2 \ldots s_n$ and the find a path in the graph.

   (b) Solve the problem by dynamic programming.

   You can assume that the only words are

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   a at me meet mid midnight night we will
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7. (20) Let $G = (V, E)$ be a simple graph. An independent set is a subset of vertices's V' such that for any pair of vertices's x and y in V', there are no edges between x and y. Show that the problem

   Is there independent set V' of G with size ($|V'|$) at least k?

   is in $\mathcal{NP}$ by giving a non-deterministic polynomial time algorithm for the problem.