Linear Algebra Midterm 2010.11.17

(Turn over leaf for more questions)

1. Suppose

$$Q = c \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}.$$

   (a) (10%) Determine c so that Q is orthogonal;

   (b) (10%) Project $b = (1, 1, 1, 1)^T$ to the space spanned by the first two columns of Q;

   Solution

   (a) Since each column must have unit length, $c = \frac{1}{2}$.

   (b) This can be solved by the normal equation $A^T A\bar{x} = A^T b$ with

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ -1 & -1 \\ -1 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

   The projection point is $A\bar{x}$.

2. (20%) Find the inverse of $n \times n$ matrix A with entries

$$a_{ij} = \begin{cases} n, & i = j \\ -1, & i \neq j \end{cases}$$

   Solution

$$a_{ij}^{-1} = \begin{cases} \frac{2}{n+1}, & i = j \\ \frac{1}{n+1}, & i \neq j \end{cases},$$

   which can be easily verified.

3. (20%) In R^3, a line L is characterized by two parameters

$$\theta = \frac{\pi}{3}, \quad \phi = \frac{\pi}{4},$$

where θ is the angle between the line and the z-axis, and ϕ is the angle between the projection of the line on the xy-plane and the x-axis.

(a) Find the *projection points* on L of the unit vectors $(1,0,0)^T$, $(0,1,0)^T$, and $(0,0,1)^T$.

(b) Find the *projection matrix* on L by computing ll^T where l is a unit vector on L.

Solution

(a) A unit vector on L is

$$l = \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{6}}{4} \\ \frac{\sqrt{6}}{4} \\ \frac{1}{2} \end{bmatrix}$$

So

$$P_L = ll^T = \begin{bmatrix} \sin^2\theta\cos^2\phi & \sin^2\theta\sin\phi\cos\phi & \sin\theta\cos\theta\cos\phi \\ \sin^2\theta\sin\phi\cos\phi & \sin^2\theta\sin^2\phi & \sin\theta\cos\theta\sin\phi \\ \sin\theta\cos\theta\cos\phi & \sin\theta\cos\theta\sin\phi & \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{8} & \frac{3}{8} & \frac{\sqrt{6}}{8} \\ \frac{3}{8} & \frac{3}{8} & \frac{\sqrt{6}}{8} \\ \frac{\sqrt{6}}{8} & \frac{\sqrt{6}}{8} & \frac{1}{4} \end{bmatrix}$$

(b) The projection points of the unit vectors are the column vectors of P_L.

4. (20%) Let $\mathcal{S}_n$ be the set of $n \times n$ real symmetric matrices, and $\mathcal{K}_n$ be the set of $n \times n$ real skew-symmetric matrices. Let $\mathcal{M}_n$ be the set of real $n \times n$ matrices. Prove that $\mathcal{K}_2$, $\mathcal{S}_3$ and $\mathcal{M}_4$ are vector spaces and decide their dimensions.

Solution

First one needs to show that if $A, B \in \mathcal{S}_n$, then $A + B \in \mathcal{S}_n$ and $cA \in \mathcal{S}_n$. Similarly for $\mathcal{K}_n$ and $\mathcal{M}_n$. For the dimensions, we have

$$\dim\mathcal{K}_2 = 1, \quad \dim\mathcal{S}_3 = 6, \quad \dim\mathcal{M}_4 = 16.$$

5. Solve the following equation

$$\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}.$$

Solution

This is solved by Gauss elimination or other methods. Omitted.

6. (15%) Find *a set of orthogonal functions* that spans the set of real-valued functions $\mathcal{F}$, where

$$\mathcal{F} = \{a + bx + cx^2 + dx^3 + ex^4 \mid a, b, c, d, e \in \mathcal{R}, x \in [-1, 1]\}.$$

Solution

As shown in class, the first three orthogonal functions are

$$v_0(x) = 1, \quad v_1(x) = x, \quad v_2(x) = x^2 - \frac{1}{3}.$$

The next one is

$$v_3(x) = x^3 - \frac{(v_0, x^3)}{(v_0, v_0)} v_0(x) - \frac{(v_1, x^3)}{(v_1, v_1)} v_1(x) - \frac{(v_2, x^3)}{(v_2, v_2)} v_2(x) = x^3 - \frac{3}{5}x$$

followed by

$$v_4(x) = x^4 - \frac{(v_0, x^4)}{(v_0, v_0)} v_0(x) - \cdots - \frac{(v_3, x^4)}{(v_3, v_3)} v_3(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35}$$

$\{v_0(x), v_1(x), v_2(x), v_3(x), v_4(x)\}$ is the answer.