Linear Algebra Final Exam
2011/1/12
Answers without due explanation or computation only get partial scores

1. (10%) Matrices and Gaussian Elimination

   (a) Find the inverse of A, where

   $$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

   (b) Write down the 2×2 matrices that

   i. turn every vector counterclockwise through $90°$;
   ii. reflect every vector through the line $x_1 = x_2$.

2. (10%) Vector Spaces

   (a) Find the rank and the nullspace of

   $$B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

   (b) Find a basis for the space of all vectors in $\mathcal{R}^6$ with

   $$x_1 + x_2 = x_3 + x_4 = x_5 + x_6$$

3. (10%) Orthogonality

   (a) If Q is orthogonal, is the same true for Q^3?

   (b) Find the curve $y = C + D2^t$ which gives the best least-squares fit to the measurements $y = 6$ at $t = 0$, $y = 4$ at $t = 1$, and $y = 0$ at $t = 2$.

4. (10%) Determinants

   (a) Find the cofactors of

   $$\begin{bmatrix} 3 & 5 \\ 6 & 9 \end{bmatrix}, \quad \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

   (b) If P_1 is an even permutation matrix and P_2 is odd, deduce from $P_1 + P_2 = P_1(P_1^T + P_2^T)P_2$ that $\det(P_1 + P_2) = 0$.

5. (10%) Eigenvalues and Eigenvectors

(a) Find the eigenvalues, eigenvectors, and the diagonalizing matrix for

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$$

(b) If the vectors x_1 and x_2 are in the columns of S, what are the eigenvalues and eigenvectors of

$$B = S \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} S^{-1}?$$

6. (10%) Positive Definite Matrices

(a) Find the values of s that make the following matrix positive definite

$$\begin{bmatrix} s & -4 & -4 \\ -4 & s & -4 \\ -4 & -4 & s \end{bmatrix}$$

(b) If B is positive definite, show from the Rayleigh quotient that the smallest eigenvalue of $A + B$ is larger than the smallest eigenvalue of A.

7. (20%) Suppose

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Find the singular value decomposition (SVD) $A = U\Sigma V^T$ and the pseudoinverse A^+.

8. (20%) Suppose

$$A = \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}.$$

What is the limit

$$\lim_{n\to\infty} A^n?$$

9. (20%) The 4×4 Pascal matrix A is defined as

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}$$

(a) Show that the inverse of A is

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

(b) Show that A is similar to its inverse by finding a diagonal matrix D with entries 1s and -1s such that

$$A^{-1} = D^{-1}AD.$$