Linear Algebra Midterm 2009.11.18

1. (extra 30%) Solve for X in the matrix equation $AX = B$, where
$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 4 \\ -3 & 2 & -7 \end{bmatrix}. \; B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}.$$

2. (20%) Prove that the product of two upper-triangular matrices is upper-triangular.

3. (20%) In R^3, find the **projection matrix** to the line with direction
$$\theta = \frac{\pi}{4}, \;\; \phi = \frac{\pi}{6},$$
where θ is the angle between the line and the z-axis, and ϕ is the angle between the projection of the line on the xy-plane and the x-axis. (hint: the projection of the basis vectors)

4. (20%) Let $\mathcal{S}_n$ be the set of $n \times n$ real *symmetric matrices*, and $\mathcal{K}_n$ be the set of $n \times n$ real *skew-symmetric matrices*. Let $\mathcal{M}_n$ be the set of real $n \times n$ matrices.
   (a) Is $\mathcal{M}_4$ a vector space? Explain.
   (b) What are the dimensions of $\mathcal{K}_4$, $\mathcal{S}_4$ and $\mathcal{M}_4$? Explain.

5. (20%) Define the **inner-product** in $\mathcal{M}_4$ as
$$(A, B) = \sum_{i,j=1}^{n} a_{ij} b_{ij}.$$
   (a) Is $\mathcal{K}_4$ orthogonal to $\mathcal{S}_4$? Explain.
   (b) Are $\mathcal{K}_4$, $\mathcal{S}_4$ orthogonal complements? Explain.

6. (20%) Let $\mathcal{F}$ be the set of functions defined on the interval $[-1, 1]$ and spanned by
$$\{1, x, x^2, x^3\}.$$
For example, $f(x) = 1 + 2x \in \mathcal{F}$. Let the inner product of two functions be defined by the integral of their product. Find a set of **orthogonal** functions that also spans F. (hint: Gram-Schmidt)