線性代數期末考　2014.01.08

1. Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}.$$

2. Factor the following matrix into $\mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1}$

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

3. Lucas started with $L_0 = 2$ and $L_1 = 1$. The rule $L_{k+2} = L_{k+1} + L_k$ is the same, so $\mathbf{A}$ is still Fibonacci matrix. Add its eigenvectors $\mathbf{x}_1 + \mathbf{x}_2$:

$$\begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} + \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} L_1 \\ L_0 \end{bmatrix}.$$

Compute the Lukas number L_{10} and λ_1^{10}.

4. From this general solution to $\frac{d\mathbf{u}}{dt} = \mathbf{A}\mathbf{u}$, find $\mathbf{A}$:

$$\mathbf{u}(t) = c_1 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

5. Compute $\mathbf{A}^H\mathbf{A}$ and $\mathbf{A}\mathbf{A}^H$:

$$\mathbf{A} = \begin{bmatrix} i & 1 & i \\ 1 & i & i \end{bmatrix}.$$

6. If the transformation T is a reflection across the $45°$ line in the plane, find its matrix with respect to the standard basis $v_1 = [1\ 0], v_2 = [0\ 1]$, and also with respect to $V_1 = [1\ 1], V_2 = [1\ -1]$.

7. (20%) Compute $\mathbf{A}^T\mathbf{A}$ and $\mathbf{A}\mathbf{A}^T$, their eigenvalues and eigenvectors, for

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Multiply the three matrices $\mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$ to recover $\mathbf{A}$.

8. Decide between maximum, minimum, or saddle point for the following cases

   (a) $F = -1 + 4(e^x - x) - 5x \sin y + 6y^2$ at the point $(0, 0)$

   (b) $F = (x^2 - 2x) \cos y$ at the point $(1, \pi)$

9. Find the (generalized) eigenvalues and eigenvectors of $\mathbf{Ax} = \lambda \mathbf{Mx}$:

$$\begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix} \mathbf{x} = \lambda \frac{1}{18} \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \mathbf{x}.$$