Introduction to Probability Final Exam

1. (10%) A smoker mathematician carries one matchbox in his left pocket and one matchbox in his right pocket. Initially, both boxes have n matches. Each time he wants to light a cigarette, he selects with equal probability to use the left or right matchbox. What is the PMF of the number of matches in the remaining matchbox when he finds the matchbox he chooses is empty?

2. (10%) Suppose that X has the PDF

$$f_X(x) = \frac{\lambda}{2} e^{-\lambda|x|}, \quad \lambda > 0.$$

   Verify that $f_X(x)$ is a valid PDF and find the mean and variance of X.

3. (10%) An absent-minded professor schedules two student appointments for the same time. The first student arrives on time while the second is late for 5 minutes. Suppose the duration of an appointment is exponential with mean 30 minutes. What is the expected value of the time between the arrival of the first student and the departure of the second student?

4. (10%) Alice looks for her term paper in her filing cabinet, which has several drawers. Assume she left it in drawer j with probability $p_j > 0$. The drawers are so messy that even if she selects the correct drawer, say drawer i, she can probably find the paper with probability d_i. Suppose she opens drawer i and fails to find the paper.

   (a) What is the probability that the paper is in drawer j for $j \neq i$?

   (b) What is the probability that the paper is indeed in drawer i?

5. (10%) Let X and Y be independent random variables uniformly distributed in $[0, 1]$. Find the CDF and the PDF of $|X - Y|$.

6. (10%) Let X be a random variable that takes nonnegative integer values. Suppose the moment generating function (or transform) of X is

$$M_X(s) = c \cdot \frac{3 + 4e^{2s} + 2e^{3s}}{3 - e^s},$$

   where c is a constant. Find $E[X], p_X(1)$, and $E[X|X \neq 0]$.

7. (10%) During each day, the probability that your computer crashes at least once is 0.05, independent of other days. Find the probability of at least 45 crash-free days out of 50 days by

   (a) using the normal approximation to the binomial

   (b) using the Poisson approximation to the binomial

8. (10%) Customers depart from a bookstore according to a Poisson process with rate λ per hour. Each customer buys a book with probability p, independent of everything else.

   (a) Find the distribution of the time until the first sale of a book.

   (b) Find the expected number of customers who buy a book during a particular hour.

9. (10%) Consider a Bernoulli process with probability p of success in each trial.

   (a) Relate the number of failures, say N_r, before the r^{th} success to a Pascal random variable and derive its PMF.

   (b) Find the expected value and variance of N_r.

10. (10%) A pizza store serves n different types of pizza, and is visited by K customers, where $K \sim$ **Poisson**(λ). Each customer orders a single pizza, with all types of pizza equally likely, independent of the orders of other customers. Find the expected number of different types of pizzas ordered.

11. (10%) Consider a gambler who at each gamble either wins or loses his bet with probability p and $1-p$, independent of earlier gambles. When $p > 1/2$, a popular gambling strategy is to always bet the fraction $2p-1$ of the current fortune. Compute the expected fortune after n gambles with the above strategy, starting with x units of money.

12. (10%) A twice-differentiable real-valued function $f(x)$ is called *convex* if $\frac{d^2 f}{dx^2}$ is nonnegative for all x in its domain of definition. Show that if X is a random variable and f is convex,

$$f(E[X]) \leq E\left[f(X)\right].$$