CALCULUS I (1031)

Midterm Exam

Department of Computer Science and Engineering
National Sun Yat-sen University

November 19, 2014, 13:20 ∼ 15:20

NAME: Solution Student ID No.: ____________

Instructor: ____________

General instructions:

1. Do not open this exam until you are told to begin.
2. This exam has 8 pages including this cover. There are 7 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each problem so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. No calculator can be used.
7. Do NOT use pencils but black or blue ball pens for your answers.
8. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to show how you arrived at your solution.
9. Please turn off all cell phones and pagers and remove all headphones.

Problem	1	2	3	4	5	6	7	**Total**
Points	10	10	10	20	10	10	30	100
Score								

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No partial credit section. (Prob.1 and 2)

No explanation is necessary. No need to show your work.

1. (10%) Find the limits in the following problems. (2% for each)

(1) $\lim_{x \to 0} \frac{\sin 4x}{x} = \mathbf{4}$.

(2) $\lim_{x \to 0} (1+x)^{1/x} = \mathbf{e}$.

(3) $\lim_{x \to \infty} \frac{-4x^2 + 2x - 5}{x^2} = \mathbf{-4}$.

(4) $\lim_{x \to \infty} \frac{5x^{3/2}}{4x^2 + 1} = \mathbf{0}$.

(5) $\lim_{x \to 1} e^{x-1} \sin \frac{\pi x}{2} = \mathbf{1}$.

2. (10%) Find dy/dx in the following problems. (2% for each)

(1) $y = x^{-2} - 2e^x$, $\frac{dy}{dx} = \mathbf{\frac{-2}{x^3} - 2e^x}$.

(2) $y = x^2 - \frac{1}{2}\cos x$, $\frac{dy}{dx} = \mathbf{2x + \frac{1}{2}\sin x}$.

(3) $y = (x^3 - 2)^2$, $\frac{dy}{dx} = \mathbf{6x^2(x^3 - 2)}$.

(4) $y = 2^{3x}$, $\frac{dy}{dx} = \mathbf{(3\ln 2)2^{3x}}$.

(5) $y^2 = \ln x$, $\frac{dy}{dx} = \mathbf{\frac{1}{2xy}}$.

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3. (10%) Find the value of $(f^{-1})'(1)$ if $f(x) = x + \cos x$. (Hint: $f(0) = 1$)

[Sol .1]

$$\because (f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))}$$

$$f(0) = 0 + \cos 0 = 1 \quad \Rightarrow \quad f^{-1}(1) = 0$$

$$f'(x) = 1 - \sin x \ . \ f'(0) = 1$$

$$(f^{-1})'(1) = \frac{1}{f'(0)} = 1$$

[Sol .2]

$$y = f^{-1}(x) \quad \Rightarrow \quad x = y + \cos y$$

$$x' = y' - (\sin y)y' \quad \Rightarrow \quad 1 = y'(1 - \sin y)$$

$$y' = \frac{1}{1 - \sin y}$$

$$x = 1 \quad \Rightarrow \quad y = 0 \text{ (Solution of } 1 = y + \cos y\text{)}$$

$$\therefore (f^{-1})(1) = y'(1) = \frac{1}{1 - \sin 0} = 1$$

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4. (20%) Find the first derivatives (dy/dx) of the functions:

(a) $y = x^{\sin x}$. (10%)

(b) $\ln xy = e^{x+y}$. (10%)

(a). $\ln y = \sin x \cdot \ln x$

$$\frac{1}{y} \cdot y' = \cos x \cdot \ln x + \sin x \cdot \frac{1}{x}$$

$$\therefore \frac{dy}{dx} = \left(x^{\sin x}\right)\left(\cos x \cdot \ln x + \frac{1}{x} \cdot \sin x\right)$$

(b). $\ln x + \ln y = e^{x+y}$

$$\frac{1}{x} + \frac{1}{y} \cdot y' = e^{x+y} \cdot (1 + y')$$

$$\therefore \frac{dy}{dx} = \frac{y(xe^{x+y} - 1)}{x(1 - ye^{x+y})}$$

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5. (10%) Use the Squeeze Theorem to find the limit:

$$\lim_{x\to\infty} \frac{\sin x}{x}$$

$\because -1 \leqslant \sin x \leqslant 1$

$\therefore -\frac{1}{x} \leqslant \frac{\sin x}{x} \leqslant \frac{1}{x}$

$\because \lim_{x\to\infty} (-\frac{1}{x}) = \lim_{x\to\infty} \frac{1}{x} = 0$

$\therefore \lim_{x\to\infty} \frac{\sin x}{x} = 0$

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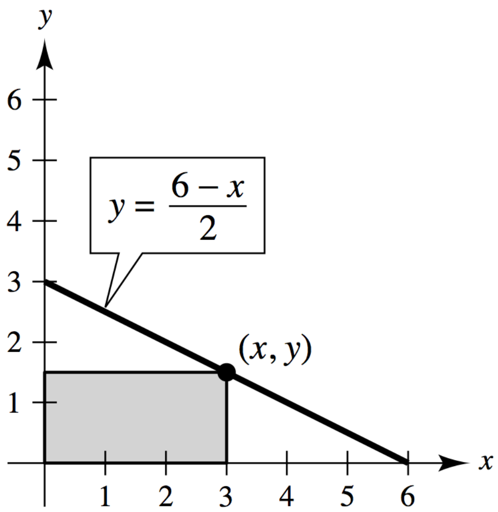
6. (10%) A rectangle is bounded by the x- and y -axes and the graph of $y = (6-x)/2$ (see figure). What length and width should the rectangle have so that its area is a maximum?



$$y = \frac{6-x}{2}$$

$$A = xy = x\left(\frac{6-x}{2}\right) = \frac{1}{2}(6x - x^2)$$

$$\frac{dA}{dx} = \frac{1}{2}(6-2x) = 0$$

$$\therefore \left.\frac{dA}{dx}\right\rfloor_{x=3} = 0$$

$$\left.\frac{d^2A}{dx^2}\right\rfloor_{x=3} = -1 < 0$$

$$\therefore \text{ when } x = 3\,, y = \frac{3}{2}\ , \text{ A is a maximum.}$$

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7. (30%) Let $f(x) = xe^x$.

(a) Find the x-intercept of f. (2%)

(b) Find the y-intercept of f. (2%)

(c) Is the function f even, odd, or neither? Justify your answer. (4%)

(d) Find the horizontal asymptotes of f. (10%)

(Hint: if $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to\infty} g(x) = \infty$, *then* $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$.)

(e) Find the intervals of increase and decrease of f. (4%)

(f) Find the local maximum and minimum value(s) of f (if any). (4%)

(g) Find the intervals of concavity of f. (4%)

[Sol]

(a). $f(x) = 0$, $xe^x = 0 \Rightarrow x = 0$, $\therefore (0, 0)$

(b). $x = 0$, $f(0) = 0$, , $\Rightarrow \therefore (0, 0)$

(c). $f(-1) = -e^{-1}$, $f(1) = e \Rightarrow f(-1) \neq f(1)$, $f(-1) \neq -f(1)$

$\therefore$ neither even nor odd.

(d). $\lim_{x\to\infty} xe^x = [\infty.\infty] = \infty$

$$\lim_{x\to-\infty} xe^x = \lim_{x\to-\infty} \frac{x}{e^{-x}} = \lim_{x\to-\infty} \frac{1}{-e^{-x}} = 0$$

$\therefore$ $y = 0$ is the horizontal asymptote.

(e). $f'(x) = e^x + xe^x = (1+x)e^x = 0 \;\Rightarrow\; x = -1$

$\therefore f'(x) < 0 \;,\; x < -1 \;\Rightarrow\; \textit{decrease}\ (-\infty, -1)$

$\therefore f'(x) > 0 \;,\; x > -1 \;\Rightarrow\; \textit{increase}\ (-1, \infty)$

(f). $\because f'(x)$ changes from negative to positive at $x = 1$

$\therefore f(-1) = -e^{-1}$ is the local minimum.

There is no local maximum.

(g). $f''(x) = \big((1+x)e^x\big)' = (2+x)e^x$

$(2+x)e^x = 0 \;,\; x = -2$

$\therefore f''(x) < 0 \;,\; x < -2 \;\Rightarrow\;$ concave down $(-\infty, -2)$

$\therefore f''(x) > 0 \;,\; x > -2 \;\Rightarrow\;$ concave up $(-2, \infty)$

(End of this exam!)