線性代數期中考 2014.11.12

答案需有適當說明。總分最多採計 100 分。

1. (10%) Apply elimination to solve the following system of linear equations

$$\begin{aligned} u+v+w &= -2 \\ 3u+3v-w &= 6 \\ u-v+w &= -1. \end{aligned}$$

2. (10%) Which of the following matrices are guaranteed to equal $(\mathbf{A}+\mathbf{B})^2$?

   $\mathbf{A}^2+2\mathbf{AB}+\mathbf{B}^2$, $\mathbf{A}(\mathbf{A}+\mathbf{B})+\mathbf{B}(\mathbf{A}+\mathbf{B})$, $(\mathbf{A}+\mathbf{B})(\mathbf{B}+\mathbf{A})$, $\mathbf{A}^2+\mathbf{AB}+\mathbf{BA}+\mathbf{B}^2$.

3. (10%)

   (a) The matrix $((\mathbf{AB})^{-1})^T$ comes from $(\mathbf{A}^{-1})^T$ and $(\mathbf{B}^{-1})^T$. In what order?

   (b) If $\mathbf{U}$ is upper-triangular then $(\mathbf{U}^{-1})^T$ is of what shape?

4. (10%) Which of the following are subspaces of $\mathbb{R}^\infty$?

   (a) All sequences like $(x_1, x_2, \dots)$ with $x_j = 0$ from some point onward.

   (b) All decreasing sequences: $x_{j+1} \le x_j$ for all j.

   (c) All convergent sequences: that x_j have a limit as $j \to \infty$.

   (d) All arithmetic progressions: $x_{j+1} - x_j$ is the same for all j.

   (e) All geometric progressions: $\frac{x_{j+1}}{x_j}$ is the same for all j.

5. (10%) The cosine space $\mathbf{F}_3$ contains all combinations

$$y(x) = A\cos x + B\cos 2x + C\cos 3x.$$

   Find a basis for the subspace that has $y(0) = 0$.

6. (10%) Suppose $\mathbb{T}$ transposes every 2×2 matrix $\mathbf{M}$. This is a linear transformation. However, there is no 2×2 matrix $\mathbf{A}$ such that

$$\mathbf{AM} = \mathbf{M}^T.$$

   Resolve the contradiction.

7. (10%) Find a vector in both column spaces $\mathcal{C}(\mathbf{A})$ and $\mathcal{C}(\mathbf{B})$:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 5 & 4 \\ 6 & 3 \\ 5 & 1 \end{bmatrix}.$$

This will be a vector $\mathbf{Ax}$ and also $\mathbf{B}\hat{\mathbf{x}}$. (hint: make use of the matrix $[\mathbf{A}\ \mathbf{B}]$)

8. (10%) Suppose $\mathbf{A}$ is 3 by 4, $\mathbf{B}$ is 4 by 5, and $\mathbf{AB} = \mathbf{0}$. Prove

$$\text{rank}(\mathbf{A}) + \text{rank}(\mathbf{B}) \leq 4.$$

9. (10%) What multiple of $\mathbf{a} = (1, 1, 1)$ is closest to the point $\mathbf{b} = (2, 4, 4)$? Find also the point closest to $\mathbf{a}$ on the line through $\mathbf{b}$.

10. (10%) For the closest parabola $b = C + Dt + Et^2$ to the points

$$(b, t): \quad (0, 0),\ (8, 1),\ (8, 3),\ (20, 4),$$

write the equations $\mathbf{Ax} = \mathbf{b}$ in 3 unknowns $\mathbf{x} = (C, D, E)$. Set up the normal equation

$$\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}.$$

11. (10%) Find an orthonormal set $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ for which $\mathbf{q}_1, \mathbf{q}_2$ span the column space of

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}.$$

Which fundamental subspace of $\mathbf{A}$ contains $\mathbf{q}_3$?