Probability Final Exam 2014/06/18

$$X \sim \textbf{uniform}(a, b), \quad f_X(x) = u(x; a, b) \triangleq \frac{1}{b-a}, \quad a \leq x \leq b$$
$$T \sim \textbf{exponential}(\lambda), \quad f_T(t) = e(t; \lambda) \triangleq \lambda e^{-\lambda t}, \quad t \geq 0$$
$$Z \sim \textbf{Poisson}(\beta), \quad p_Z(k) = p(k; \beta) \triangleq e^{-\beta}\frac{\beta^k}{k!}, \quad k = 0, 1, \ldots$$
$$G \sim \textbf{geometric}(p), \quad p_G(k) = g(k; p) \triangleq (1-p)^{k-1}p, \quad k = 1, 2, \ldots$$

All answers must be supported by arguments or computations! Good luck.

1. (10%) John Fast is driving from Boston to New York, a distance of 180 miles at a constant speed, whose value is uniformly distributed between 60 and 90 miles per hour. What is the PDF of the duration of the trip?

2. (10%) Two archers shoot at a target. The distance of each shot from the center of the target is uniformly distributed from 0 to 1, independent of the other shot. What is the PDF of the winning shot?

3. (10%) We are given a biased coin and we are told that because of manufacturing defects, the probability of heads, denoted by Y, is itself random, with a uniform distribution over the interval $[0.4, 0.7]$. We toss the coin a fixed number n of times. What is the expected value of the number of heads X?

4. (10%) Jane visits a number of bookstores, looking for *Great Expectations*. Any given bookstore carries the book with probability 0.2, independent of the others. In a bookstore visited, Jane spends a random amount of time, exponentially distributed with parameter 3, until she either finds the book or she determines that the bookstore does not carry it. What is the PDF of the total time spent in bookstores until Jane buys a copy?

5. (10%) A computer executes 2 types of tasks, priority and non-priority, in discrete time units (slots). A priority task arrives with probability 0.6 at the beginning of each slot, independent of other slots, and requires one full slot to complete. A non-priority task is always available and is executed at a given slot if no priority task is available. A slot is called *busy* if a priority task is executed within this slot. Otherwise it is called *idle*. A busy period is a string of busy slots flanked by idle slots.

Let B and I be the lengths of the first busy period and the first idle period, respectively. What are the distributions of B and I?

6. (10%) You get emails according to a Poisson process at a rate of 2 messages per hour. You check your emails every hour. What is the probability of finding 0 or 1 new messages? ($e \approx 2.72$)

7. (10%) A professor has 2 umbrellas that he uses when commuting between home and office. It it rains he takes an available umbrella with him. If it is not raining, he does not take an umbrella. Suppose the probability of rain is 0.6 every time he commutes. What is the probability that he gets wet during a commute?

8. (20%) Alice is taking a course. In each week, she is either up-to-date (state 1) or falling behind (state 2). If she is up-to-date in a given week, the probability that she will be up-to-date in the next week is 0.5. If she is falling behind in a given week, the probability that she will be falling behind in the next week is 0.8. What is the mean first passage time to state 1, starting from state 2? What is the mean recurrence time to state 1?

9. (10%) A gambler wins \$1 at each round with probability 1/3 and loses \$1 with probability 2/3. Different rounds are independent. Beginning with \$2, he plays until he either accumulates \$4 or loses all money. What is the probability that he loses all his money?