線性代數期末考 2013.01.09

1. (10 pt.) Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}.$$

2. (10 pt.) Suppose permutation matrix $\mathbf{P}$ takes $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$ to $\begin{bmatrix} 5 \\ 4 \\ 1 \\ 2 \\ 3 \end{bmatrix}$.

   (a) What does $\mathbf{P}^2$ do to $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$?

   (b) What does $\mathbf{P}^{-1}$ do to $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$?

3. (10 pt.) Find the eigenvalues of the following matrix

$$\mathbf{A} = \begin{bmatrix} 0.2 & 0.4 & 0.3 \\ 0.4 & 0.2 & 0.3 \\ 0.4 & 0.4 & 0.4 \end{bmatrix}.$$

   Note that $\mathbf{a}_1 + \mathbf{a}_2 = 2\mathbf{a}_3$ and that $\mathbf{A}$ is Markovian.

4. (10 pt.) Let F_n be the $1, 1, -1$ tridiagonal matrix of size $n \times n$:

$$F_n = \begin{vmatrix} 1 & -1 & & & \\ 1 & 1 & -1 & & \\ & 1 & 1 & -1 & \\ & & \ddots & \ddots & \ddots \\ & & & 1 & 1 \end{vmatrix}.$$

   By **expanding in cofactors along row 1**, show that

$$F_n = F_{n-1} + F_{n-2}.$$

5. (10 pt.) The characteristic polynomial of $\mathbf{A} = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is

$$\lambda^2 - (a+d)\lambda + (ad - bc).$$

   By direct substitution, show that

$$\mathbf{A}^2 - (a+d)\mathbf{A} + (ad-bc)\mathbf{I} = \mathbf{0}.$$

6. (**20** pt.) Find the eigenvalues and eigenvectors of the Markovian matrices

$$\mathbf{A} = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix} \quad \text{and} \quad \mathbf{A}^\infty = \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix}$$

7. (10 pt.) Diagonalize the unitary matrix

$$\mathbf{V} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ 1+i & 1 \end{bmatrix}$$

to reach $\mathbf{V} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^H$.

8. (10 pt.) Find $e^{\mathbf{A}t}$ where

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

9. (10 pt.) If $\mathbf{P}_1$ is an even permutation matrix and $\mathbf{P}_2$ is odd, deduce from $\mathbf{P}_1 + \mathbf{P}_2 = \mathbf{P}_1(\mathbf{P}_1^T + \mathbf{P}_2^T)\mathbf{P}_2$ that $|\mathbf{P}_1 + \mathbf{P}_2| = 0$.

10. (5 pt.) If $\mathbf{K}^H = -\mathbf{K}$ (skew-Hermitian), the eigenvalues are imaginary and the eigenvectors are orthogonal. How do you know that $\mathbf{K} - \mathbf{I}$ is invertible?