Probability Final Exam 2013/6/14

1. (10%) A machine, once in production mode, operates continuously until an alarm signal is generated. The time up to an alarm signal is an exponential random variable with mean 5. Subsequent to an alarm signal, the machine is tested for an random time which is exponential with mean 1/3. The test results are positive with probability 1/4, in which case the machine returns to production mode, or negative with probability 3/4, in which case the machine is taken for repair. The repair takes an exponential random time with mean 1/2. Please write down the transition probability matrix of the embedded DTMC.

2. (10%) (as above) What is the transition probability matrix of the δ-DTMC, neglecting $o(\delta)$ terms?

3. (10%) Alice is taking a course in linear algebra. In each week, she is either up-to-date or falling behind. If she is up-to-date in a given week, the probability that she will be up-to-date in the next week is 3/4. If she is behind in a given week, the probability that she will be behind in the next week is 4/5. Suppose she is up-to-date this week. What is the expected time until she is up-to-date again (i.e. the mean recurrence time)?

4. (10%) A gambler wins $1 at each round with probability 2/5 and loses $1 with probability 3/5. Different rounds are independent. Beginning with $3, he plays until he either accumulates $5 or loses all money. What is the probability that he loses all his money?

5. (10%) A computer executes 2 types of tasks, priority and non-priority, in discrete time units (slots). A priority task arrives with probability 0.2 at the beginning of each slot, independent of other slots, and requires one full slot to complete. A non-priority task is always available and is executed at a given slot if no priority task is available. A slot is called *busy* if a priority task is executed within this slot. Otherwise it is called *idle*. A busy period is a string of busy slots flanked by idle slots.

   Let Z be the number of slots after the first slot of the first busy period up to and including the first subsequent idle slot. What is the mean of Z?

6. (10%) You get emails according to a Poisson process at a rate of $\lambda = 0.1$ messages per minute. You check your emails every hour. What is the probability of finding 2 new messages?

7. (10%) People with letters to mail arrive at the post office according to a Poisson process with rate 0.5 arrivals per minute. People with packages to mail arrive at the post office according to a Poisson process with rate 0.2 arrivals per minute, independent of the people with letters to mail. What is the probability that the first customer wants to mail a letter?

8. (10%) Buses arrive at a station deterministically: on the hour, and 20 minutes after the hour. A tourist shows up at the station at a random time (uniformly distributed) and wait for a bus. What is the expected value of waiting time?

9. (10%) A machine can be either working or broken down on a given day.

   - When working, it breaks down in the next day with probability 0.2.
   - When broken, it is working in the next day with probability 0.7.
   - However, if it stays down for 3 days in a row, it will be replaced by a new working machine in the next day.

   Draw the transition probability graph for this machine.

10. (10%) A professor has 2 umbrellas that he uses when commuting between home and office. It it rains he takes an available umbrella with him. If it is not raining, he does not take an umbrella. Suppose the probability of rain is 0.3 every time he commutes. What is the probability that he gets wet during a commute?