1. (10) Let f and g be two functions from integers to integers. State the definition of "$f(n)$ is $O(g(n))$" and then prove that $(2n+3)^2$ is $O(n^2)$ by giving the constants n_0 and c in the definition of O-notation.

2. (20) Suppose that the time complexity of a problem $\mathcal{P}$ has been proved to have a lower bound $\Omega(\sqrt{n})$, and that the best known algorithm for solving this problem runs in linear time. Give a short comment for each of the following research projects. Your comments should be one of the following: (A) a valuable project, (B) an impossible project, (C) a valueless project.
   In addition to the short comments, you also need to explain the reasons why you give such a comment.
   (a) Design an algorithm to solve problem $\mathcal{P}$ in $O(\log n)$ time.
   (b) Design an algorithm to solve problem $\mathcal{P}$ in $O(\log^2 n)$ time.
   (c) Proof that the lower bound of the time complexity of the problem $\mathcal{P}$ is $\Omega(\log n)$.
   (d) Proof that the lower bound of time complexity of the the problem $\mathcal{P}$ is $\Omega(n)$.

3. (15) Let a and b be two n-bit integers, n is very large.
   (a) Is it possible to design an algorithm for computing a^2 which is asymptotically faster than computing $a \times b$?
   (b) Is it possible to design an algorithm for computing $a \times b$ which is asymptotically faster than computing a^2?
   (c) Supposed that someone has shown that a program for computing a^2 takes less time than the best program for computing $a \times b$. How do you explain this?

4. (15) Suppose that a computer can only do addition (+) and arithmetic shift (<< or >>). Write C code to compute the following statements efficiently.
   (a) $y = 10x$.
   (b) $y = 15x$.
   (c) Suppose that the computer can also do subtraction (−), in addition to addition and shift. Show how to compute $y = 15x$ more efficiently.

5. (20) Let p be a prime, and a be an element in $\mathbf{Z}_p^*$. Consider the two sets $A = \{1, 2, \ldots, p-1\}$ and $B = \{1a, 2a, \ldots, (p-1)a\}$. Note that "$xy$" means "$xy \bmod p$".
   (a) Show that the two sets A and B are equal.
   (b) Show that if p is composite, than the two sets may not be equal.

6. (20) Let $x_1, x_2, \ldots, x_n$ be a sequence of n integers. It is known that the sequence is *unimodal*, which means that $x_1, x_2, \ldots, x_m$ is strictly increasing, and $x_{m+1}, x_{m+2}, \ldots, x_n$ is strictly deceasing, $1 \le m \le n$. Design an $o(n)$ time algorithm to find the maximum element x_m. How many comparisons does your algorithm need in the worst case?

7. (10) After a test, the scores of n students are stored in an array $A[1..n]$. Assume that all scores are positive integers. Give a linear time algorithm to rearrange the n scores stored in the array so that all the scores greater than or equal to 60 appears before the scores less than 60.

8. (10) Let a and b be two positive constants. Show that $T(n)$ is linear if
$$T(n) \le \begin{cases} a & \text{if } n \le 20, \\ T(n/5) + T(7n/10) + bn & \text{otherwise.} \end{cases}$$

9. (10) Design a method to find the median of 5 integers with only 6 comparisons in the worst case.