Introduction to Probability Midterm 2012/4/23

1. (10%) Romeo and Juliet have a date at a given time, and each will arrive at the meeting place with a delay between 0 and 1 hours, with all pairs of delays being equally likely. The first to arrive will wait for only 15 minutes. What is the probability that they will meet?

2. (5%) Design team C and design team N are separately asked to design a new product within a month. The probability that team C is successful is 2/3, the probability that team N is successful is 1/2, and the probability that at least one team is successful is 3/4. Assuming that exactly one successful design is produced, what is the probability that it was designed by team N?

3. (5%) You enter a chess tournament where your probability of winning is 0.3 against half the players, 0.4 against a quarter of the players and 0.5 against the remaining players. You play a game against a randomly chosen opponent. What is your probability of winning?

4. (10%) If an aircraft is present in a certain area, a radar detects it and generates an alarm with probability 0.99. If an aircraft is not present in that area, a radar generates an alarm with probability 0.1. Assume that an aircraft is present with probability 0.05. Given an alarm is generated, what is the probability that an aircraft is present in the area?

5. (10%) Let A and B be events with $P(A) > 0$ and $P(B) > 0$. We say that B suggests A if $P(A|B) > P(A)$.

   (a) Show that B suggests A if and only if A suggests B.

   (b) Assume that $P(B^c) > 0$. Show that B suggests A if and only if B^c does not suggest A.

6. (10%) Let X and Y be independent standard normal random variables. The pair (X, Y) can be described in polar coordinates in terms of random variables $R \geq 0$ and $\Theta \in [0, 2\pi]$, so that

$$X = R\cos\Theta, \quad Y = R\sin\Theta.$$

   Show that Θ is uniformly distributed in $[0, 2\pi]$, and R has the (Rayleigh) PDF

$$f_R(r) = re^{-r^2/2}, \quad r > 0.$$

7. (10%) Consider the following two-sided exponential PDF

$$f_X(x) = \begin{cases} p\lambda e^{-\lambda x}, & x \geq 0, \\ (1-p)\lambda e^{\lambda x}, & x < 0. \end{cases}$$

Find the mean and the variance of X.

8. (10%) A coin has probability of heads equal to p. It is tossed successively and independently until back-to-back heads or back-to-back tails appear. Find the expected value of the number of tosses using the total expectation theorem.

9. (10%) A binary signal is transmitted, and we are given that $P(S = 1) = p$ and $P(S = -1) = 1 - p$. The received signal is $Y = N + S$, where N is a standard normal random variable independent of S. What is the probability that $S = 1$, as a function of the observed value y of Y?

10. (5%) Let X, Y be described by a uniform PDF on the unit square. What is the joint CDF?

11. (5%) Suppose $X \sim$ **uniform**(a, b). What is the variance of X?

12. (10%) Suppose that X and Y are independent, identically distributed, geometric random variables with parameter p. Show that

$$P(X = i | X + Y = n) = \frac{1}{n-1}, \quad i = 1, \ldots, n-1.$$

13. (10%) Suppose $X \sim$ **Poisson**(λ), that is,

$$p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots$$

Find the mean and the variance of X?