機率學期末考 請說明推導或運算過程

1. (30%) Consider continuous random variables X and Y with the following joint probability density function (PDF)

$$f_{X,Y}(x,y) = ce^{-q(x,y)},$$

where the exponent term $q(x,y)$ is given by

$$q(x,y) = \frac{\frac{x^2}{\sigma_x^2} - 2\rho\frac{xy}{\sigma_x\sigma_y} + \frac{y^2}{\sigma_y^2}}{2(1-\rho^2)}.$$

σ_x, σ_y, c are positive constants, and ρ is a constant with $-1 < \rho < 1$.

   (a) Show that X and Y are zero-mean, with variances σ_x^2 and σ_y^2.

   (b) Show that the correlation coefficient of X and Y is ρ.

   (c) Show that the estimation error $E[X|Y] - X$ is normal with mean zero and variance $(1-\rho^2)\sigma_x^2$.

2. (10%) What is the moment generating function (MGF) of a binomial random variable

$$B \sim \textbf{binomial}(n,p)?$$

3. (30%) Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with zero mean and variance σ^2. Let the MGF be $M_X(s)$ for an interval $-d < s < d,\ d > 0$. Let

$$Z_n = \frac{\sum_{k=1}^{n} X_k}{\sigma\sqrt{n}}.$$

   (a) Show that the MGF of Z_n satisfies

$$M_{Z_n}(s) = \left(M_X\left(\frac{s}{\sigma\sqrt{n}}\right)\right)^n.$$

   (b) Let the second-order Taylor expansion of $M_X(s)$ be

$$M_X(s) = a + bs + cs^2 + o(s^2).$$

   Find a, b, and c in terms of σ^2.

(c) Show that

$$\lim_{n\to\infty} M_{Z_n}(s) = e^{s^2/2}.$$

4. (10%) Let Y_k be the time of the k^{th} arrival in a Poisson process with rate λ. Show that for all $y > 0$,

$$\sum_{k=1}^{\infty} f_{Y_k}(y) = \lambda.$$

5. (10%) Let T be an exponential random variable with parameter ν. Find the probability mass function (PMF) of the number of arrivals in the time interval $[0, T]$ in a Poisson process with rate λ.

6. (10%) Consider a discrete-time Markov chain. Show that if we add up the first $m-1$ balance equations,

$$\pi_j = \sum_{k=1}^{m} \pi_k p_{kj}, \quad j = 1, 2, \ldots, m-1,$$

we obtain the last equation

$$\pi_m = \sum_{k=1}^{m} \pi_k p_{km}.$$

7. (20%) 畫出轉移機率圖 (佳芬題)

(a) Gambler's ruin: Gabor wants to win 5 dollars, the probability of winning a round is 0.5;

(b) Random walk with reflecting barriers: there are 4 positions ($1, 2, 3, 4$), the probability to the right is 0.3;

(c) Queueing: the buffer size is 4, the arrival probability is 0.2, the departure probability is 0.4;

(d) Professor with 2 umbrellas: the probability of raining is 0.1.