Linear Algebra Final 2012.1.4

1. (10%) Show that
$$(A^{-1}+B^{-1})^{-1} = A(A+B)^{-1}B.$$

2. (10%) Suppose S is spanned by the vectors $(1,2,2,3)$ and $(1,3,3,2)$. Find two vectors that span $S^{\perp}$.

3. (10%) Prove that the trace of
$$P = \frac{aa^T}{a^Ta},$$
where a is a non-zero vector, is always equal to 1.

4. (10%) A is a matrix of rank 1
$$A = \begin{bmatrix}1\\2\\1\end{bmatrix}\begin{bmatrix}2 & 1 & 2\end{bmatrix}.$$
Find the eigenvalues and eigenvectors of A.

5. (10%) Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of A. Substitute $A = S\Lambda S^{-1}$ into the product
$$(A-\lambda_1 I)(A-\lambda_2 I)\dots(A-\lambda_n I)$$
to show that the result is the zero matrix.

6. (10%) If A is a Markov matrix, show that the sum of the components of Ax equals the sum of the components of x.

7. (10%) Find the lengths of
$$u = \begin{pmatrix}1+i & 1-i & 1+2i\end{pmatrix}, \quad v = \begin{pmatrix}i & i & i\end{pmatrix}.$$
Also find $u^H v$ and $v^H u$.

8. (10%) Prove that the inverse of a Hermitian matrix is Hermitian.

9. (10%) Explain why $A+I$ is never similar to A.

10. (10%) Find the matrix A for the quadratic form

$$f(x_1, x_2) = 3(x_1 + 2x_2)^2 + 4x_2^2.$$

Factor A into LDL^T.

11. (10%) Draw the ellipse

$$x^2 + xy + y^2 = 1.$$

Find the half-lengths of its axes.

12. (10%) Find $A = U\Sigma V^T$ if A has orthogonal columns $w_1, \ldots, w_n$ of lengths $\sigma_1, \ldots, \sigma_n$.