Department of Computer Science and Engineering
National Sun Yat-sen University
Second Semester of 2023 PhD Qualifying Exam

Subject: Probability

Fill out the indexed blanks (5 points/blank) in the answer sheet, e.g. $\underline{\text{(1)} \quad 50}$.

1. Suppose the probability of heads of a biased coin is $\frac{1}{3}$. Let A be the number of flips until the first appearance of a Head immediately followed by a Tail. Let B be the number of flips until the first appearance of 2 consecutive Heads.

$$\mathbf{E}[A] = \underline{\text{(1)}}$$

$$\mathbf{E}[B] = \underline{\text{(2)}}$$

2. Romeo and Juliet have a date, for which each arrives with a delay between 0 and 1 hour, with all pairs of delays being equally likely. Romeo waits at most 30 minutes and Juliet waits at most 20 minutes.

   (a) The probability that they meet is $\underline{\text{(3)}}$.

   (b) Suppose they meet. The probability that Romeo arrives first is $\underline{\text{(4)}}$.

3. 5 persons mix their hats in a box and each picks one hat randomly. Let N be the number of person(s) picking own hat(s).

$$\mathbf{E}[N] = \underline{\text{(5)}}$$

$$\mathbf{E}[N^2] = \underline{\text{(6)}}$$

4. Fill in $>, =,$ or $<$. Consider $X \sim$ **Uni**$(1, 4)$ (uniform in $(1, 4)$).

$$\mathbf{E}[\log X] \; \underline{\text{(7)}} \; \log(\mathbf{E}[X])$$

$$\mathbf{E}\left[X^2\right] \; \underline{\text{(8)}} \; (\mathbf{E}[X])^2$$

5. $X \sim$ **Uni**$(0, 2)$ and $Y \sim$ **Uni**$(0, 2)$ are independent. Consider $S = X + Y$ and $Z = \max(X, Y)$.

   (a) $\mathbf{var}(S) = \underline{\text{(9)}}$.

   (b) $f_Z(z)$ (the probability density function of Z) at $z = 1$ is $\underline{\text{(10)}}$.

6. $X \sim \mathbf{Uni}(-1, 2)$ and $Y = g(X)$ where

$$g(x) = \begin{cases} 1, & \text{if } x \leq 1 \\ 2, & \text{if } x > 1 \end{cases}$$

   (a) $\mathbf{E}[Y] = \underline{(11)}$.

   (b) $\mathbf{E}[\mathbf{E}[X|Y]] = \underline{(12)}$.

7. X is a continuous random variable and $Y = F(X)$ where

$$F(x) = P(X \leq x)$$

   (a) $P(Y > 1) = \underline{(13)}$.

   (b) $P(-\log_e Y > 1) = \underline{(14)}$.

8. $X \sim \mathbf{Exp}(2)$ and $Y \sim \mathbf{Exp}(2)$ are independent exponential random variables with parameter 2. Consider $T_1 = \min(X, Y)$ and $T_2 = \max(X, Y)$.

$$\mathbf{E}[T_1] = \underline{(15)}$$

$$\mathbf{var}(T_2) = \underline{(16)}$$

9. $X \sim \mathbf{Uni}(1, 2)$ and $Y \sim \mathbf{Uni}(1, 2)$ are independent, and $Z = |X - Y|$.

$$\mathbf{E}[Z|Y] = \underline{(17)}$$

$$\mathbf{E}[Z] = \underline{(18)}$$

10. $X \sim \mathbf{Uni}(0, 1)$ and $Y \sim \mathbf{Uni}(0, 1)$ are independent.

$$P\left(\frac{Y}{3X} < 1\right) = \underline{(19)}$$

$$P\left(\frac{2Y}{X} < 1\right) = \underline{(20)}$$