Department of Computer Science and Engineering
National Sun Yat-sen University
Second Semester of 2024 PhD Qualifying Exam

Subject: **Probability**

Instruction: Fill out the indexed blanks in the answer sheet, e.g. ① 50 .

Questions

1. Let $X \sim \textbf{Uni}(0,2)$.
$$\mathbf{E}\left[\log\left(\frac{X}{2-X}\right)\right] = \text{①}$$
$$\mathbf{E}[\min(X, 2-X)] = \text{②}$$

2. Let $Z \sim \mathcal{N}(0,1)$ with CDF $P(Z \le z) = \Phi(z)$.
$$\mathbf{E}\left[Z^4\right] = \text{③}$$
$$P(1 < Z^2 < 4) = \text{④}$$

3. Let $X \sim \textbf{Bin}\left(10, \frac{1}{2}\right)$ and $Y \sim \textbf{Bin}\left(9, \frac{1}{2}\right)$ be independent.
$$P(X > Y) = \text{⑤}$$
$$P(\sqrt{X} \le 2) = \text{⑥}$$

4. A tennis game is in *deuce*. In this game, the server wins a point with probability 3/4. Let N be the number of points to be played further to finish this game, and I be the indicator for the event that the server wins this game eventually.
$$\mathbf{E}[I] = \text{⑦}$$
$$\mathbf{E}[N] = \text{⑧}$$

5. Let $U \sim \textbf{Uni}(0,1)$ and $V \sim \textbf{Uni}(0,1)$ be independent and $M = \max(U,V)$.
$$\mathbf{E}[M] = \text{⑨}$$
$$\mathbf{E}[M|V] = \text{⑩}$$

6. Let X and Y be independent $\textbf{Exp}(1)$. Define $M = \max(X,Y)$ and $L = \min(X,Y)$.
$$\mathbf{E}[L] = \text{⑪}$$
$$\mathbf{E}[M|L] = \text{⑫}$$

7. Let Z and W be independent $\mathcal{N}(0,1)$. Define $M = \max(Z,W)$ and $L = \min(Z,W)$.
$$\mathbf{E}[M-L] = \text{⑬}$$
$$|\mathbf{E}[M]| - |\mathbf{E}[L]| = \text{⑭}$$

8. X is a continuous random variable and $Y = F(X)$ where $F(x) = P(X \le x)$.

$$P\left(X \le \frac{1}{3}\right) = \underline{\textcircled{15}}$$

$$P\left(Y \le \frac{1}{3}\right) = \underline{\textcircled{16}}$$

9. 7 persons mix their hats in a box and each picks one hat randomly. Let H be the number of person(s) picking own hat(s).

$$\mathbf{E}[H] = \underline{\textcircled{17}}$$

$$\mathbf{E}[H^2] = \underline{\textcircled{18}}$$

10. Let Q and R be independent **Poi**(1) and $S = Q + R$.

$$P(S = 10) = \underline{\textcircled{19}}$$

$$P(Q = 5 | S = 10) = \underline{\textcircled{20}}$$

Reference

$X \sim \textbf{Uni}[a, b] \;\Rightarrow\; p(x) = \dfrac{1}{b - a + 1}$ for $x = a, a + 1, \ldots, b$

$X \sim \textbf{Ber}(p) \;\Rightarrow\; p(x) = p^x(1 - p)^{1-x}$ for $x = 0, 1$

$X \sim \textbf{Geo}(p) \;\Rightarrow\; p(x) = p(1 - p)^{x-1}$ for $x = 1, 2, \ldots$

$X \sim \textbf{Poi}(\lambda) \;\Rightarrow\; p(x) = e^{-\lambda}\dfrac{\lambda^x}{x!}$ for $x = 0, 1, \ldots$

$X \sim \textbf{Bin}(n, p) \;\Rightarrow\; p(x) = \dbinom{n}{x} p^x(1 - p)^{n-x}$ for $x = 0, 1, \ldots, n$

$X \sim \textbf{Uni}(a, b) \;\Rightarrow\; f(x) = \dfrac{1}{b - a}$ for $a < x < b$

$X \sim \textbf{Exp}(\lambda) \;\Rightarrow\; f(x) = \lambda e^{-\lambda x}$ for $x > 0$

$X \sim \mathcal{N}(\mu, \sigma^2) \;\Rightarrow\; f(x) = \dfrac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

$\sqrt{e} \doteq 1.65,\ \sqrt{\pi} \doteq 1.77,\ \log 2 \doteq 0.69$

$\Phi(0.5) \doteq 0.69,\ \Phi(1.0) \doteq 0.84,\ \Phi(1.5) \doteq 0.93$

$\Phi(2.0) \doteq 0.98,\ \Phi(2.5) \doteq 0.994,\ \Phi(3.0) \doteq 0.999$