Algorithms

There are 7 problems in this test. Do any 5 problems.

1. (20%) Give asymptotic lower and upper bounds for each of the following recurrences. Make your bounds as tight as possible and prove that the bounds are correct.

   (a) $T(1) = 0$, $T(n) = T(\lceil 9n/10 \rceil) + n$, for $n > 1$.

   (b) $T(1) = 0$, $T(n) = T(\lceil \sqrt{n} \rceil) + 1$, for $n > 1$.

2. (20%) The subgraph isomorphism problem is to determine whether an undirected graph contains a subgraph which is isomorphic to another undirected graph. The satisfiability problem is to determine whether a Boolean formula has a truth assignment to satisfy the formula. The clique problem is to determine whether a graph contains a complete subgraph of size k. The Hamiltonian problem is to determine whether an undirected graph contains a Hamiltonian cycle. Assume that the satisfiability problem, the clique problem and the Hamiltonian problem are all known to be NP-complete. Prove that the subgraph isomorphism problem is NP-complete.

3. (20%) Let $a_1, a_2, \ldots, a_n$ be a sequence of n distinct integers sorted in ascending order. Binary search can be used to find the index i such that $a_i = k$ for a given input k, or report that no such integer exits in the sequence, in $O(\log n)$ time. Design efficient algorithm for each of the following cases. The time complexity of your algorithms must be better than the time complexity of binary search.

   (a) k is relatively small, for example, $k = o(\log n)$.

   (b) k is relatively large, that is, k would appear near the end of the sequence, if it exists.

4. (20%) Let $X = x_1, x_2, \ldots, x_n$ be a sequence of n numbers. A subsequence of X is a sequence obtained from X by deleting some elements. Design an $O(n^2)$ algorithm to find the longest monotonically increasing subsequence of a given sequence X.

5. (20%) The tower of Hanoi problem is to move n disks from peg A to peg C using peg B as a buffer. The diameters of the n disks are all distinct, and they are initially stacked in peg A in decreasing diameter. (That is, disk with largest diameter is placed at the bottom of the stack.) The disks can only be moved one at a time, and a disk with larger diameter can never be stacked on top of a disk with smaller diameter. Describe a nonrecursive algorithm for the tower of Hanoi problem by using a stack. Analyze the size of the stack required by your algorithm.

6. (20%) Let G be a weighted graph, T be the minimum spanning tree obtained by a computer program. Design an efficient algorithm to verify that T is indeed a minimum spanning tree of G.

7. (20%) Define the terms *non-deterministic algorithm* and *probabilistic algorithm*. Give an example for each of the two types of algorithms.