Qualifying Exam: Probability

1. (20%) An absent-minded professor schedules two student appointments for the same time. The first student arrives on time while the second is late for 10 minutes. Suppose the duration of an appointment is exponential with mean 20 minutes. What is the expected value of the time between the arrival of the first student and the departure of the second student?

2. (20%) Alice looks for a paper in her filing cabinet with three drawers. Assume she left it in drawer $1, 2, 3$ with probability $0.5, 0.3, 0, 2$ respectively. The drawers are so messy that even if she selects the correct drawer, say drawer i, she can probably find the paper with probability d_i. Assume that $d_1 = 0.5, d_2 = 0.7, d_3 = 0.8$. Suppose she opens drawer 1 and fails to find the paper. What are the probabilities that the paper is in drawer 1, 2 and 3?

3. (20%) Let X and Y be independent random variables uniformly distributed in $[0, 1]$. Find the cumulative distribution function (CDF) and the probability density function (PDF) of $Z = X + Y$.

4. (20%) Three light bulbs have independent exponentially distributed lifetimes with a common mean of 300 days. What is the expected value of the time until the last bulb burns out?

5. (20%) Gambler Tom has \$500. In one game, he wins \$100 with probability 0.4, or loses \$100 with probability 0.6. Different games are assumed to be independent. He continues to gamble until either he loses all his money or he wins $\$1,000$. What is the probability that he wins?